Introduction to Differential Equations

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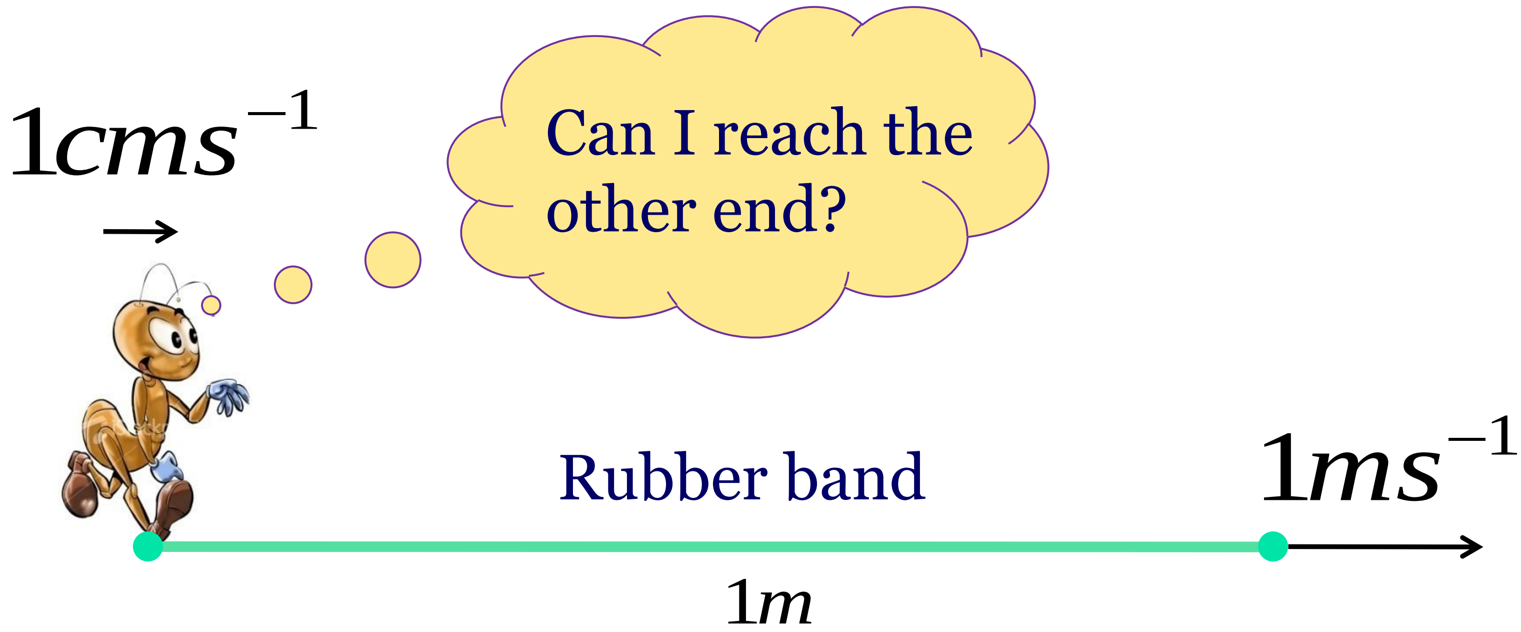
Can Antson reach the other end?



Can Antson reach the other end?

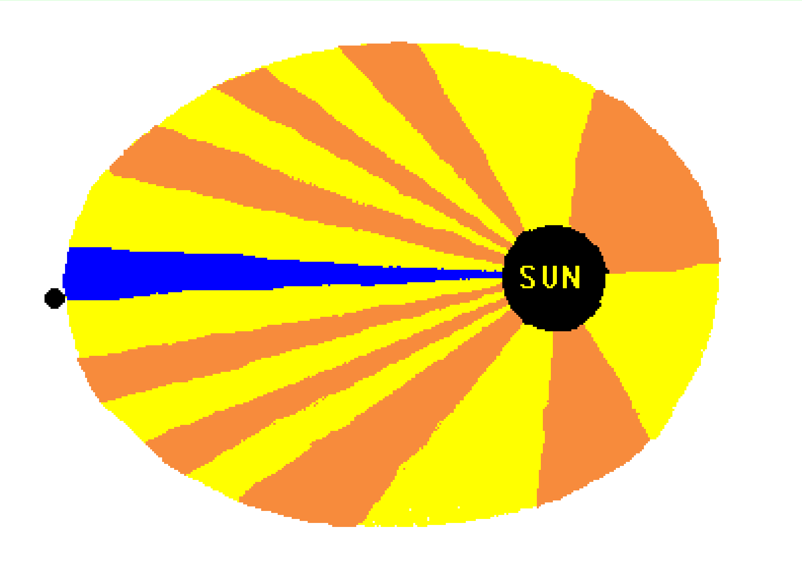
Gottfried Wilhelm Leibniz (1646-1716)

- German mathematician and philosopher
- Credited for, along with Newton, the discovery of calculus
- Invented the use of $\int$ and d.

Isaac Newton (1643-1727)

Kepler's laws of planetary motion

1. The orbit is an ellipse with the sun at one of the foci.
2. A line joining a planet and the sun sweeps out equal areas in equal time.



3. The squares of the orbital periods are directly proportional to the cubes of the semi-major axes.

Kepler's laws of planetary motion

Inverse square law	$\Longrightarrow$	$T^2 \propto R^3$
Conservation of momentum	$\Longrightarrow$	Equal area
Differential equation	$\Longrightarrow$	Elliptic orbit

Inverse square law

Centripetal force:

$$F = \frac{mv^2}{R} = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2 \propto \frac{R}{T^2}$$

Assume inverse square law

$$F \propto \frac{1}{R^2}$$

Then

$$\frac{1}{R^2} \propto \frac{R}{T^2}$$

$$T^2 \propto R^3$$

Conservation of angular momentum

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

is constant

$\Rightarrow$

Angular momentum

$$L = m\vec{r} \times \vec{v}$$
$$= mr^2\dot{\theta}$$

is constant

Elliptic orbit

Newton second law: $\dfrac{\vec{F}}{m} = \vec{a}$

$$-\frac{GM}{r^2}\hat{e}_r = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\Rightarrow \begin{cases} \ddot{r} - r\dot{\theta}^2 = -\dfrac{GM}{r^2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{cases}$$

Elliptic orbit

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$$r^2\dot{\theta} = l$$

$$\dot{\theta} = \frac{l}{r^2}$$

In fact, this is known already from conservation of angular momentum.

Elliptic orbit

$$-\frac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2$$

$$\ddot{r} - r\left(\frac{l}{r^2}\right)^2 = -\frac{GM}{r^2}$$

Therefore we need to solve

$$\ddot{r} - \frac{l^2}{r^3} = -\frac{GM}{r^2}$$

Elliptic orbit

Let $a = \dfrac{l^2}{GM}$ and $u = \dfrac{a}{r}$

$$\dot{r} = \frac{d}{dt}\left(\frac{a}{u}\right) = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{a}{u}\right) = -\frac{lu^2}{a^2}\cdot\frac{a}{u^2}u' = \frac{lu'}{a}$$

$$\ddot{r} = -\frac{d}{dt}\left(\frac{lu'}{a}\right) = -\frac{l}{a}\frac{d\theta}{dt}\frac{d}{d\theta}u' = -\frac{l^2u^2u''}{a^3}$$

Elliptic orbit

$$\ddot{r} - \frac{l^2}{r^3} = -\frac{GM}{r^2}$$

$$-\frac{l^2 u^2 u''}{a^3} - \frac{l^2 u^3}{a^3} = -\frac{l^2 u^2}{a^3}$$

The equation is simplified to

$$u'' + u = 1$$

Elliptic orbit

The general solution is

$$u'' + u = 1$$

$$u = 1 + \varepsilon \cos(\theta - \alpha)$$

$$r = \frac{a}{1 + \varepsilon \cos(\theta - \alpha)}$$

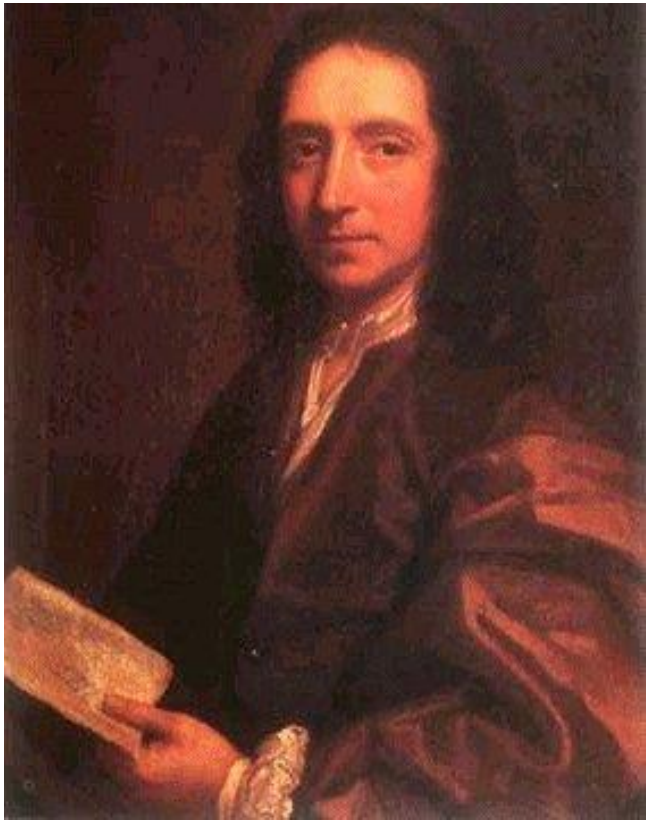
Recall:

$$u = \frac{a}{r}$$

which represents a conic curve with focus at the origin.

Edmond Halley (1656-1742)

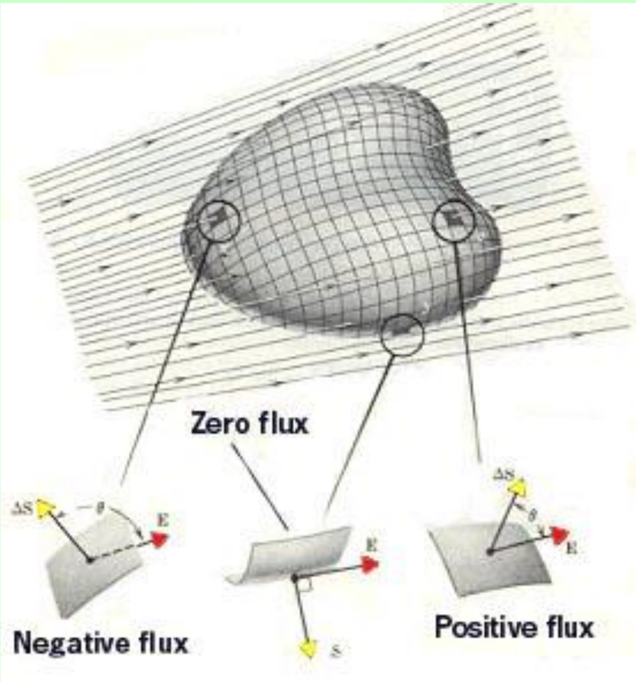
- Claim that the comet sightings of 1456, 1531, 1607 and 1682 related to the same comet.
- Predicted that the comet would return in 1758.
- The Halley's comet was seen again on 25th Dec 1758.



Electromagnetism

Gauss' law

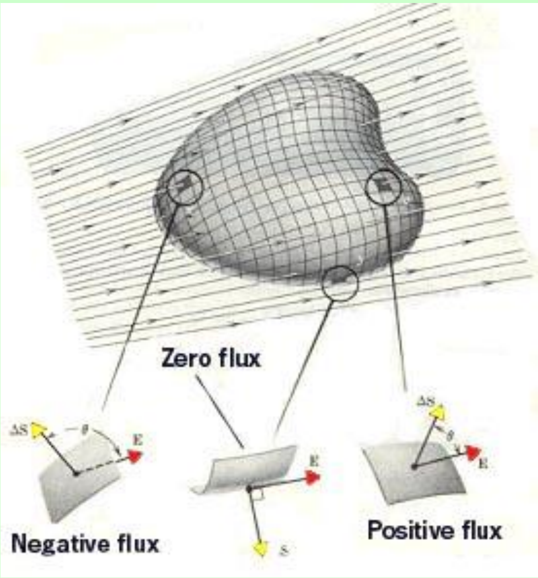
$$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$



Electromagnetism

Gauss' law for magnetism

$$\oiint_S \vec{B} \cdot d\vec{A} = 0$$



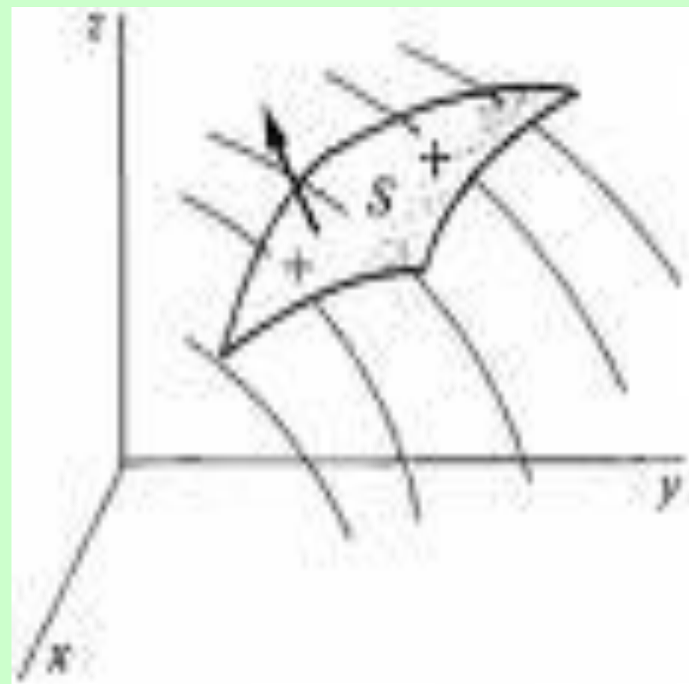
Electromagnetism

Faraday's law

$$\oint_{\partial S} \vec{E} \times d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$$

where

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A}$$



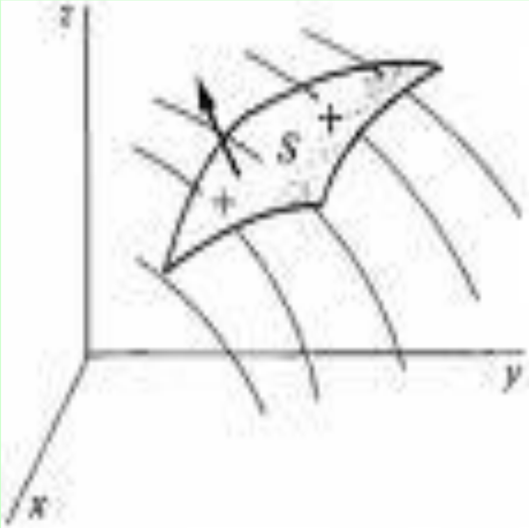
Electromagnetism

Ampere's law

$$\oint_{\partial S} \vec{B} \times d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$$

where

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{A}$$



Maxwell's equations

Name	Integral form	Differential form
Gauss' law	$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
Gauss' law	$\oiint_S \vec{B} \cdot d\vec{A} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday's law	$\oint_{\partial S} \vec{E} \times d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampere's law	$\oint_{\partial S} \vec{B} \times d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Electromagnetic wave

In vacuum, Maxwell's equations become

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Electromagnetic wave

Using the identity $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{\nabla}^2\vec{A}$

We have

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\vec{\nabla}^2\vec{E}$$

$$\vec{\nabla}\times\left(-\frac{\partial B}{\partial t}\right)=-\vec{\nabla}^2\vec{E}$$

$$-\frac{\partial}{\partial t}\left(\vec{\nabla}\times B\right)=-\vec{\nabla}^2\vec{E}$$

$$-\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial E}{\partial t}\right)=-\vec{\nabla}^2\vec{E}$$

$$\vec{\nabla}^2\vec{E}=\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

Electromagnetic wave

$$\vec{\nabla}^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

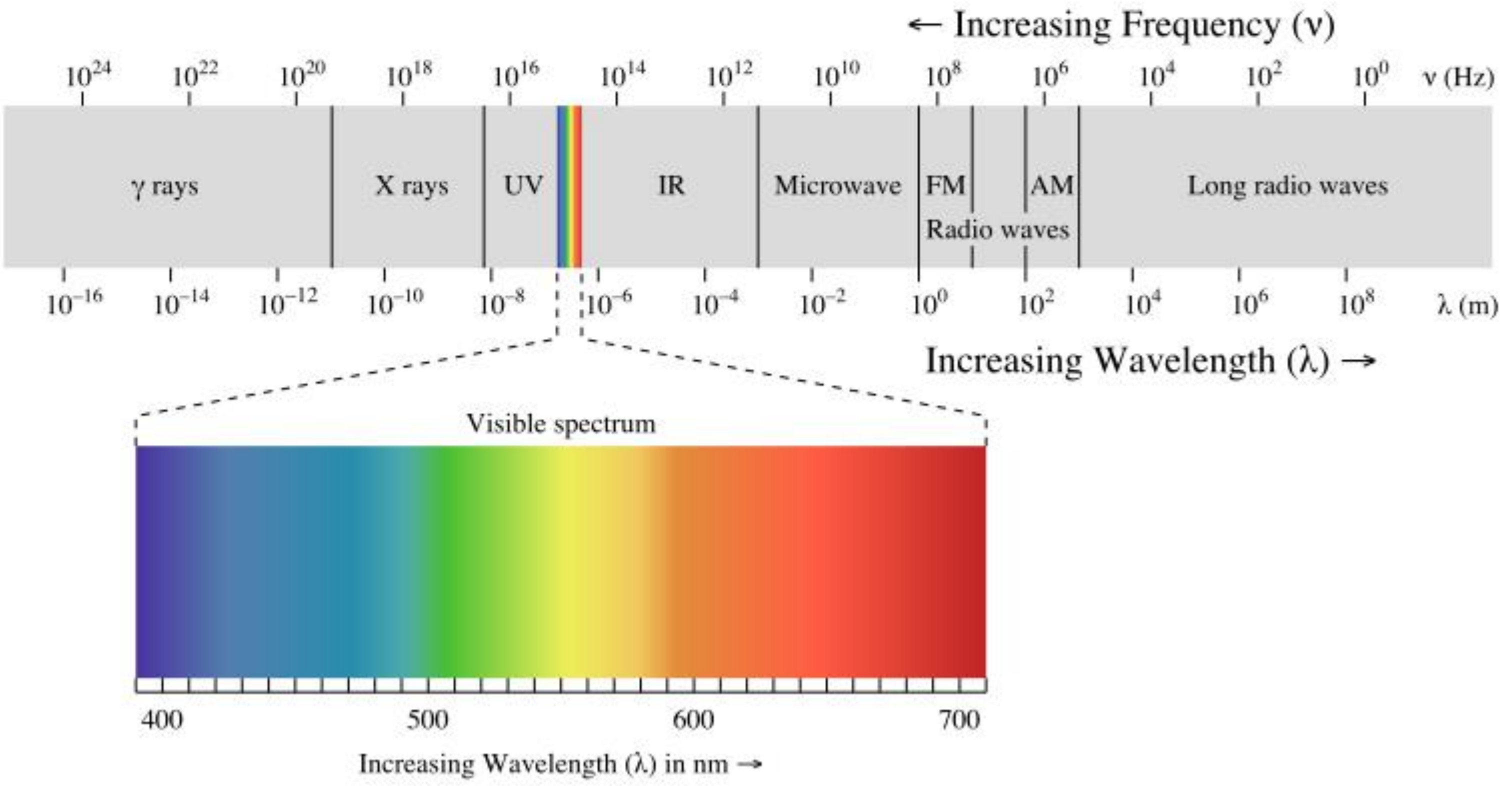
The above equation shows the existence of wave of oscillating electric and magnetic fields which travel at a speed

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 300{,}000 kms^{-1}$$

which is very close to the speed of light.

Maxwell then claimed that light is in fact electromagnetic wave.

Electromagnetic wave



Special relativity

Maxwell's equation in tensor form

$$\begin{cases} F^{\alpha\beta}{}_{,\alpha} = \dfrac{4\pi}{c} J^{\beta} \\ F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \end{cases}$$

where

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and

$$J^{\beta} = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

are the electromagnetic tensor and the 4-current.

General relativity

According to Einstein field equation, gravity is described as a curved space time caused by matter and energy.

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}$$

$R_{\alpha\beta}$: Ricci tensor

R : scalar curvature

$g_{\alpha\beta}$: metric tensor

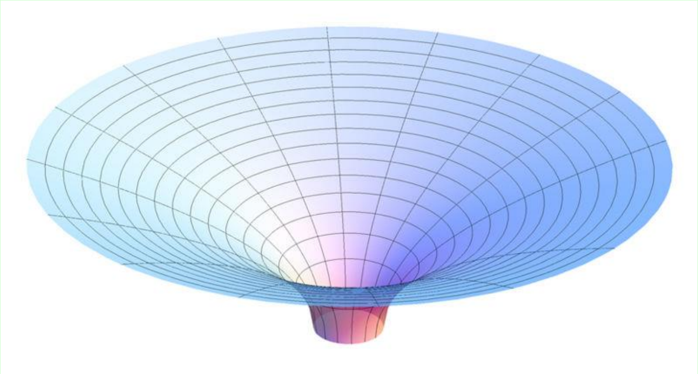
$T_{\alpha\beta}$: energy-momentum-stress tensor

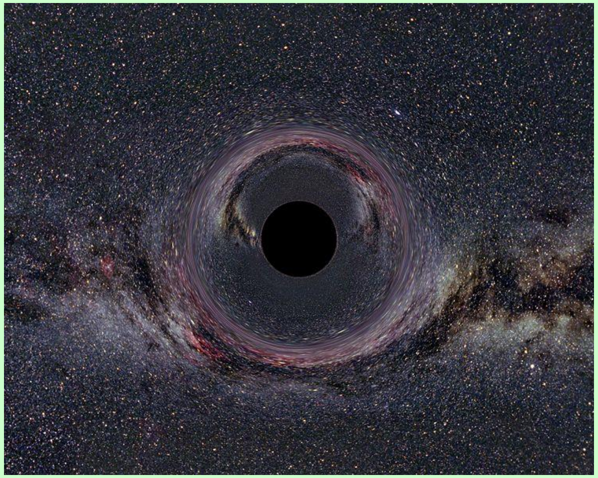
Schwarzschild black hole

A black hole with no change or angular momentum.

Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$



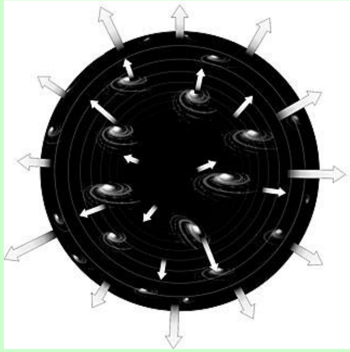


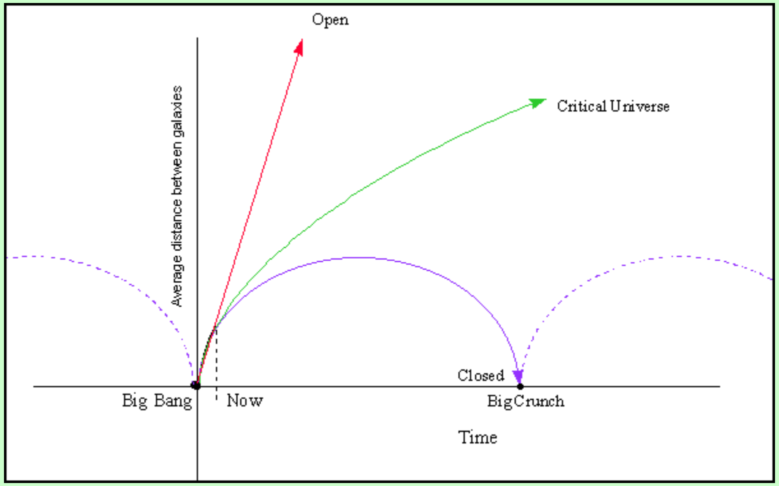
Expanding universe

Robertson-Walker metric

$$ds^2 = c^2 dt^2 - R^2(t)\left(d\chi^2 + S^2(\chi)d\Omega^2\right)$$

$$S(\chi) = \begin{cases} \sin\chi, & \text{curvature} > 0 \\ \chi, & \text{curvature} = 0 \\ \sinh\chi, & \text{curvature} < 0 \end{cases}$$





Schrödinger equation

In quantum mechanics, particles are described by wave function satisfying

$$i\frac{h}{2\pi}\frac{d\psi}{dt} = H\psi$$

where

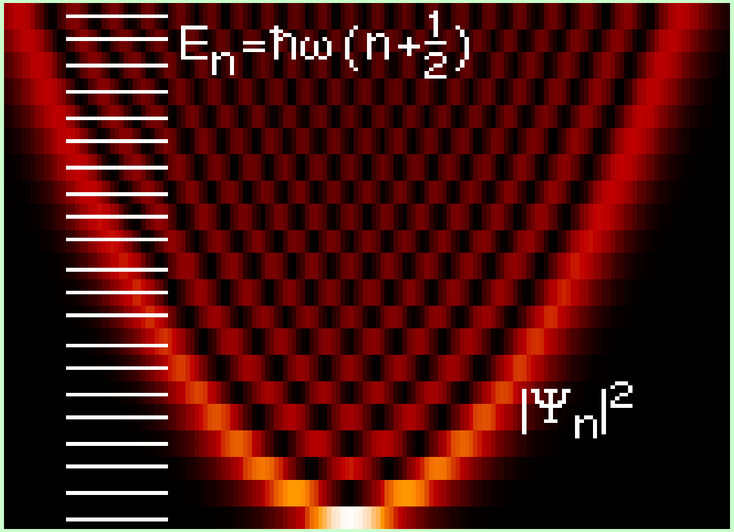
h : Planck's constant

ψ : wave function

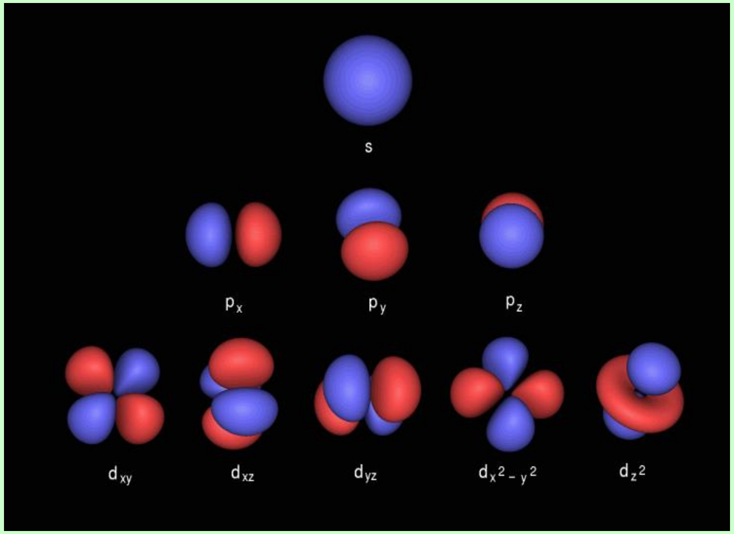
H : Hamiltonian operator

Schrödinger equation

Harmonic Oscillator



Electron orbitals



Black-Scholes' equation

Black-Scholes model the price of an option by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where V : price of the option

S : price of the underlying instrument

σ : volatility

r : constant interest rate

Calabi’s conjecture

Let $(M, g_{i\bar{j}})$ be a compact Kähler manifold. Any closed (1,1)-form which represents the first Chern class of M is the Ricci form of a metric determines the same cohomology class as $g_{i\bar{j}}$.

Calabi's conjecture

Equivalent to the existence of solution of the following complex Monge-Ampère equation

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) \det\left(g_{i\bar{j}}\right)^{-1} = \exp(F)$$

where

$$\int_M \exp(F) = Vol(M)$$

Proved by Yau Shing Tung in 1976.

Navier-Stokes equation

Navier-Stokes Equation describe the motion of viscous fluid.

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla\mathbf{p} + \mu\Delta\mathbf{v} + \mathbf{f}$$

where

$\mathbf{v}$	: velocity
ρ	: density
p	: pressure
$\mathbf{f}$	: external force

The continuity equation reads $\nabla\cdot\mathbf{v} = 0$

Poincaré's conjecture

Every compact simply-connected 3 dimensional manifold is homeomorphic to the 3 dimensional sphere.

Generalized Poincaré's conjecture

If a compact n dimensional manifold is homotopic to the n dimensional sphere, then it is homeomorphic to the n dimensional sphere.

Generalized Poincaré's conjecture

Dimension	Solver	Year	Field's Medal
1 or 2	Classical		
5 or above	Stephen Smale	1960	1966
4	Michael Freeman	1982	1986
3	Grigori Perelman	2003	2006

Ricci flow

Proved by Perelman by using Ricci flow defined by Hamilton.

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

Perelman declined both the Fields medal and the Clay Millennium Prize.

Definition

An Ordinary Differential Equation of order n is an equation of the form

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

where $y^{(n)}$ denotes the nth derivative of y.

Definition

If there are more than one independent variable and the equation involves partial derivatives, then it is called Partial Differential Equation.

Examples

First order ODE:

i) Linear equations

$$a) \quad \frac{dy}{dx} + 4y = 0$$

$$b) \quad \frac{dy}{dx} - xy = \cos x$$

ii) Bernoulli equation

$$y' + p(x)y = q(x)y^n$$

Examples

Second order ODE:

i) Linear equations

$$a)\quad y'' = 2y' - y$$

$$b)\quad y'' - x^2 y' + e^{3x} y = 2\sin x$$

ii) Non-linear equations

$$a)\quad y'' = y^2$$

$$b)\quad y'' + yy' = e^x$$

Examples

PDE:

i) Elliptic $$u_{xx} + u_{yy} = 0$$

ii) Parabolic $$u_t = u_{xx} + u_{yy}$$

ii) Hyperbolic $$u_{xx} + u_{yy} - u_{tt} = 0$$

Solution

Differential equation	Solution
$y' = y + 2$	$y = Ce^{x} - 2$
$\frac{dy}{dx} = -\frac{x^2 + xy}{3xy + y^2}$	$2x^3 y + x^2 y^2 = C$
$y'' - 3y' - 4y = 5e^{-x}$	$y = C_1 e^{4x} + C_2 e^{-x} - xe^{-x}$
$u_{xx} - 4u_{tt} = 0$	$u = \cos(2x - t)$ *

* Particular solution

IVP and BVP

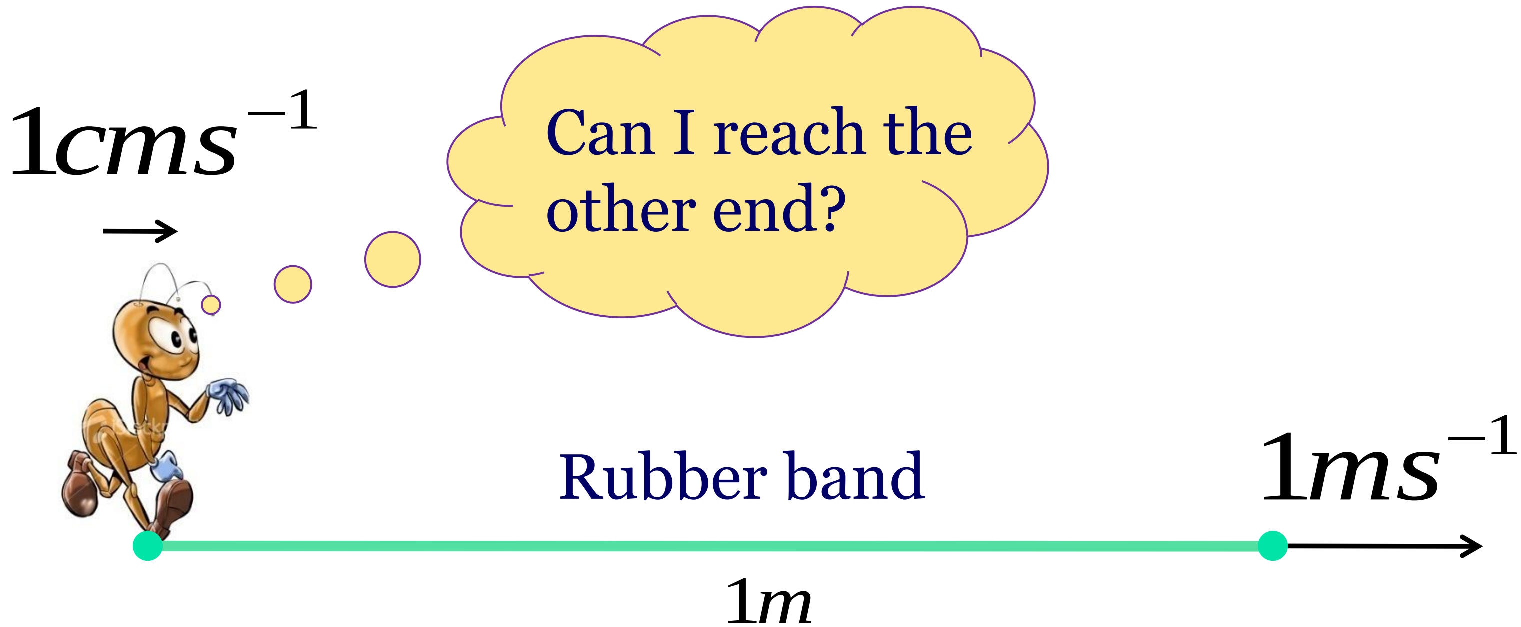
Initial value problem:

$$\begin{cases} y''-3y'+2y=\sin x, \quad x\in[0,2\pi] \\ y(0)=0,\ y'(0)=1 \end{cases}$$

Boundary value problem:

$$\begin{cases} y''-3y'+2y=\sin x, \quad x\in[0,2\pi] \\ y(0)=0,\ y(2\pi)=-2 \end{cases}$$

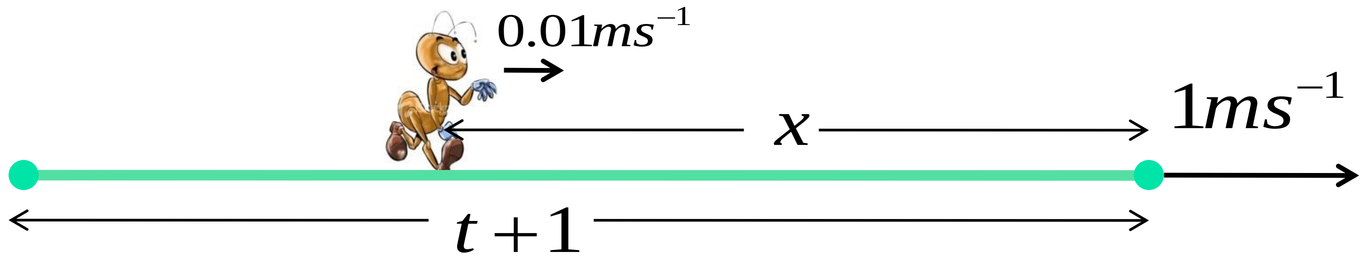
Can Antson reach the other end?



Can Antson reach the other end?

Antson can always reach the other end when $u > 0$.

When $u = 0.01$ and $v = 1$



$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{t+1} - 0.01 \\ x(0) = 1 \end{cases}$$

Sol: $x = (t+1)\left(1 - \dfrac{\ln(t+1)}{100}\right)$

$$x(t) = 0$$
$$\Rightarrow \quad \ln(t+1) = 100$$
$$\Rightarrow \quad t = e^{100} - 1 \approx 2.7 \times 10^{43}$$

It takes about

8.5×10^{35} years

First order equation

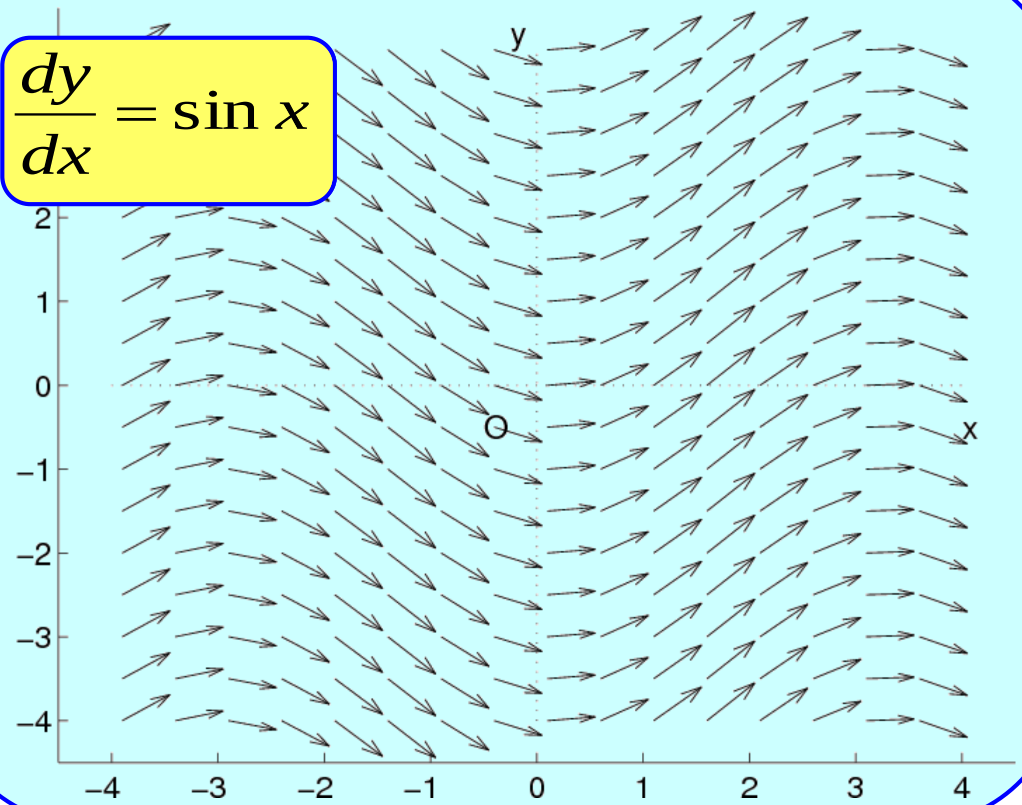
The first order ODE

$$\frac{dy}{dx} = f(x, y)$$

can be interpreted as a direction field. The integral curves are solutions of the equation.

Direction field

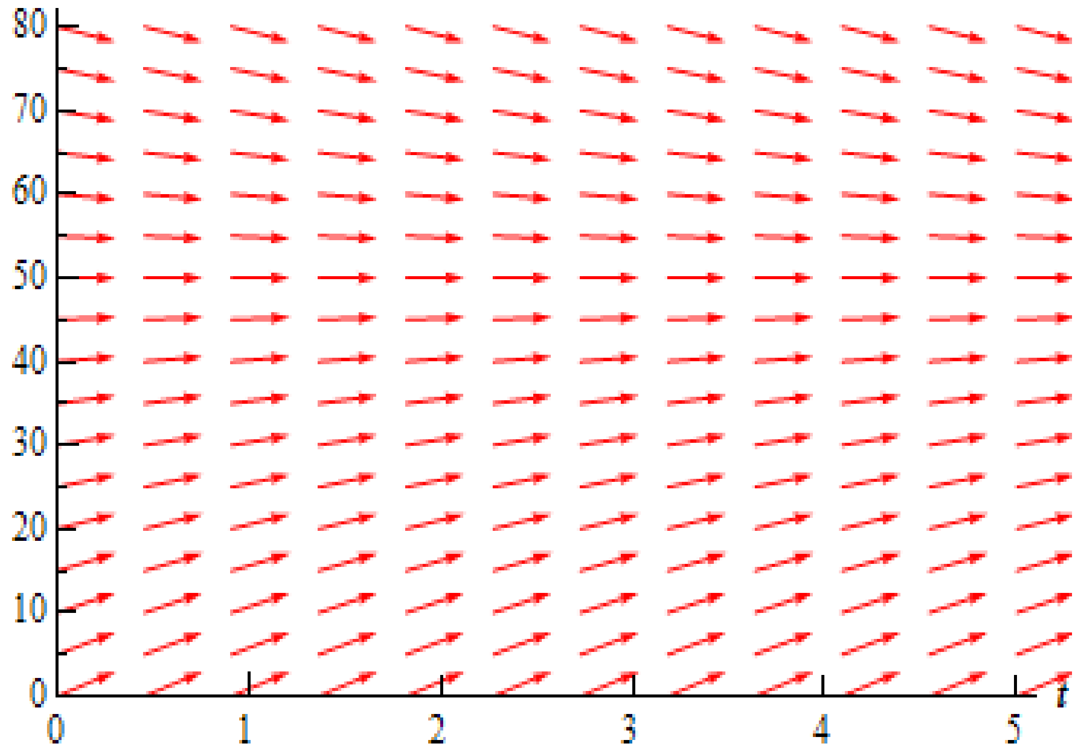
$$\frac{dy}{dx} = \sin x$$



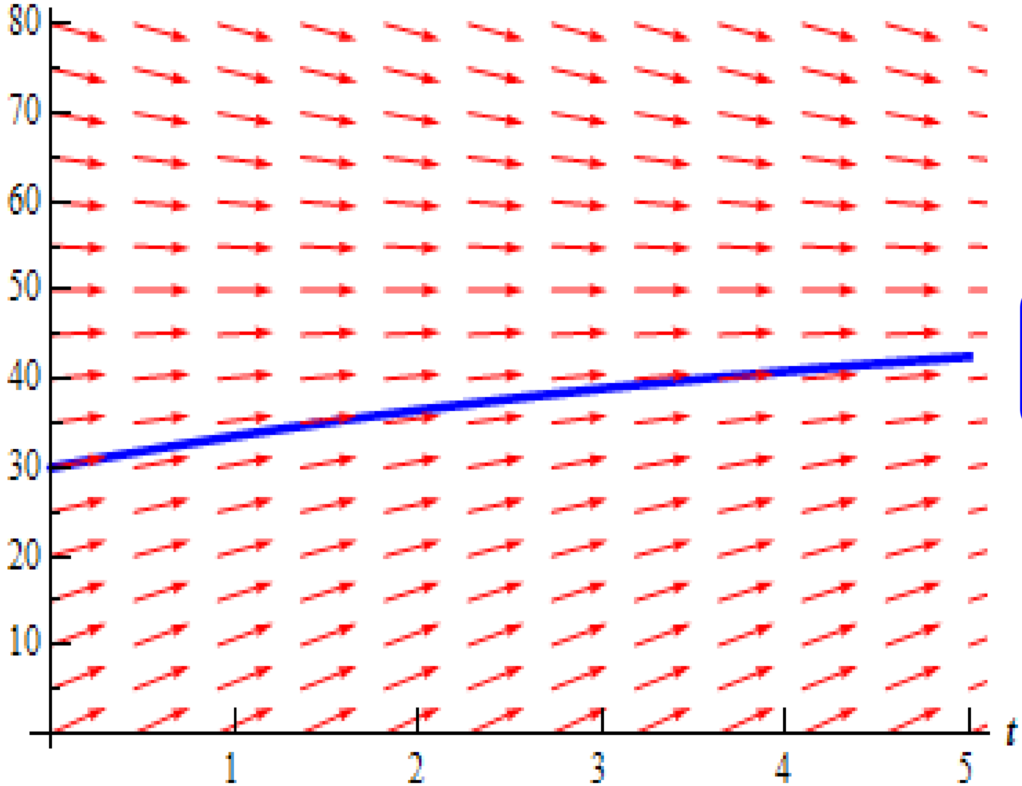
$$y = \int \sin x \, dx$$
$$= -\cos x + C$$

Direction field

$$\frac{dy}{dx} = 10 - \frac{y}{5}$$



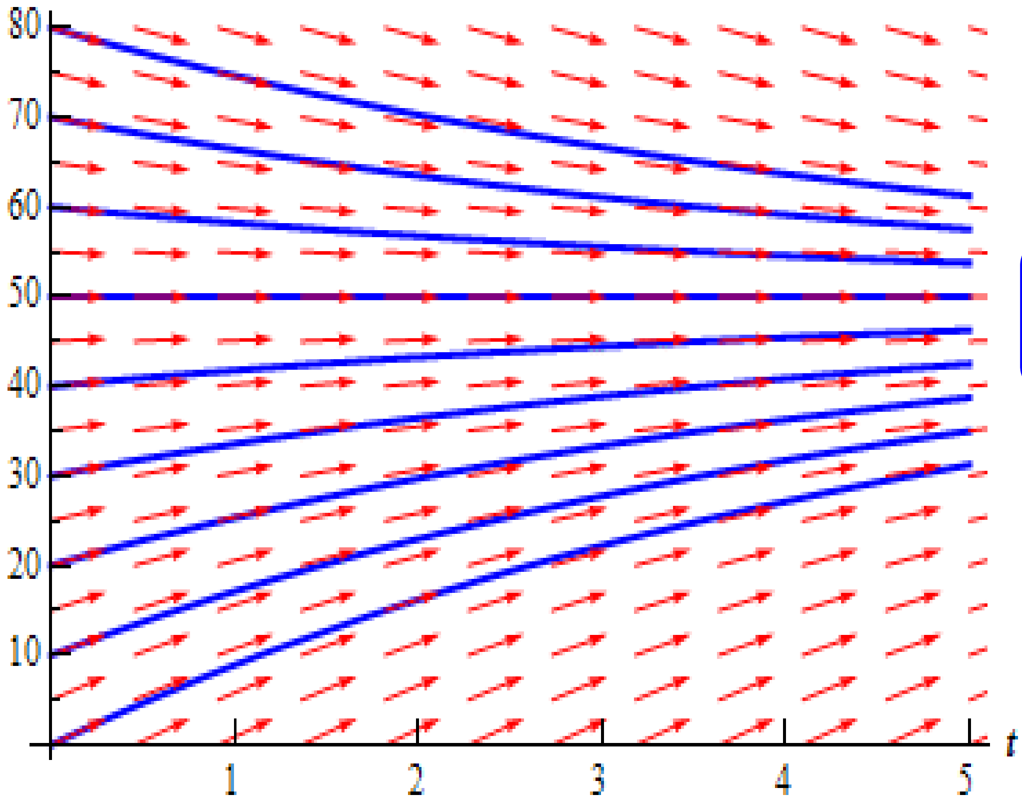
Direction field



$$\frac{dy}{dx} = 10 - \frac{y}{5}$$

$$y = 50 - 20e^{-\frac{x}{5}}$$

Direction field

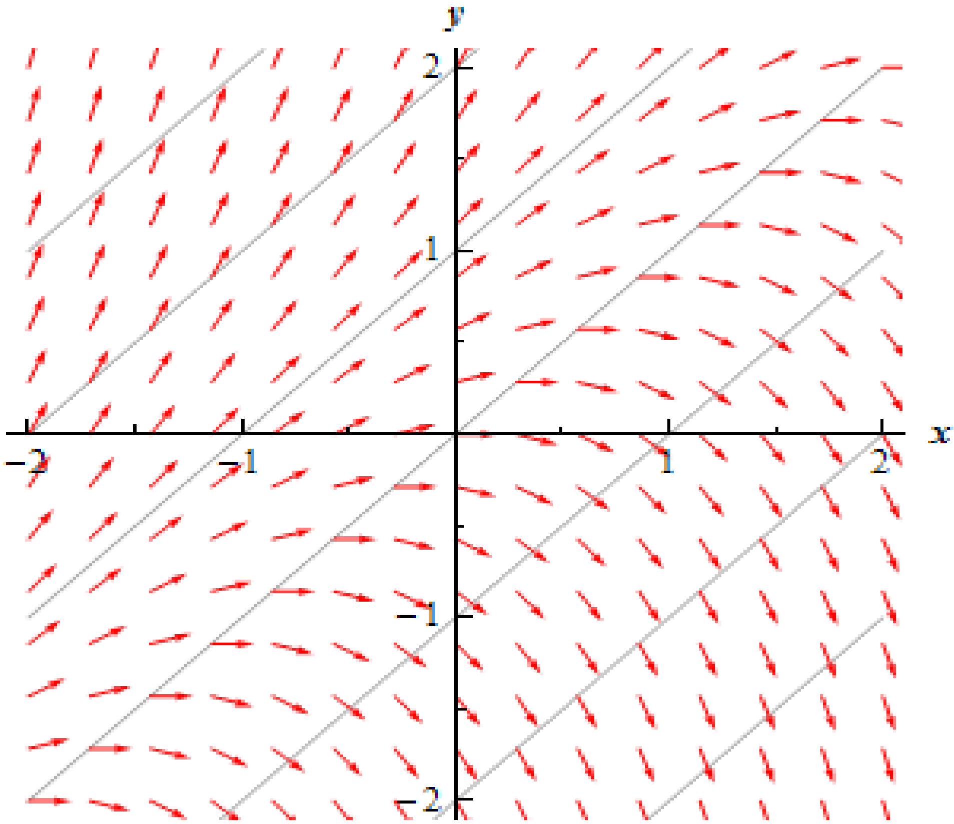


$$\frac{dy}{dx} = 10 - \frac{y}{5}$$

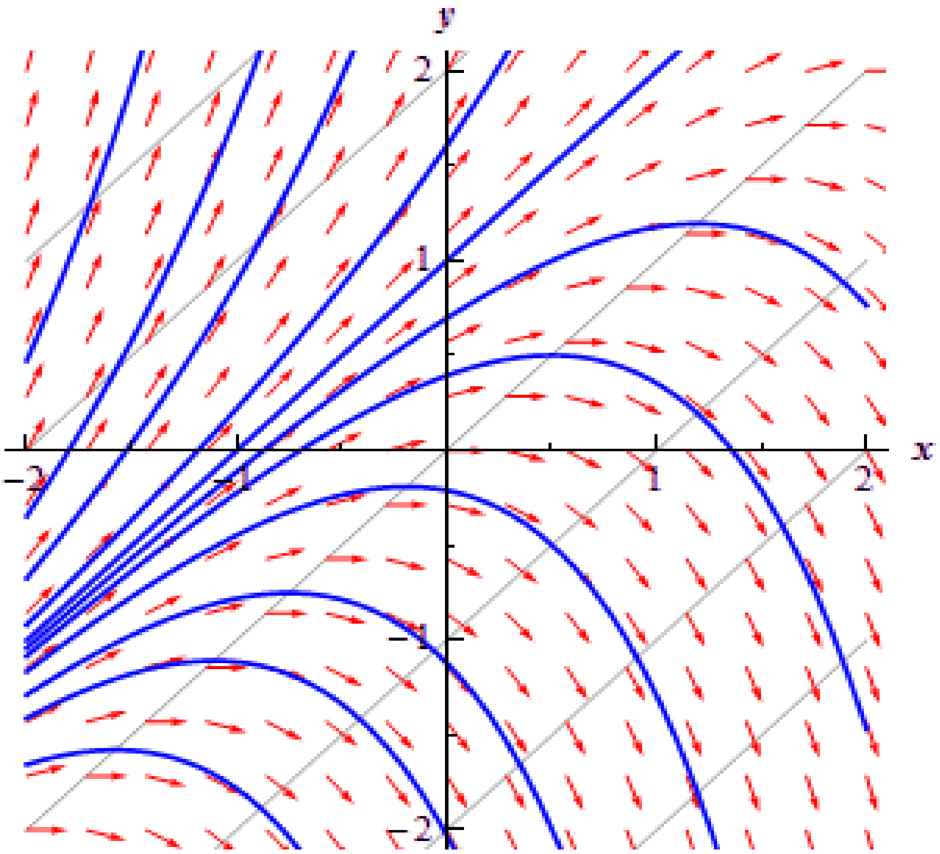
$$y = 50 - Ce^{-\frac{x}{5}}$$

Direction field

$$\frac{dy}{dx} = y - x$$



Direction field

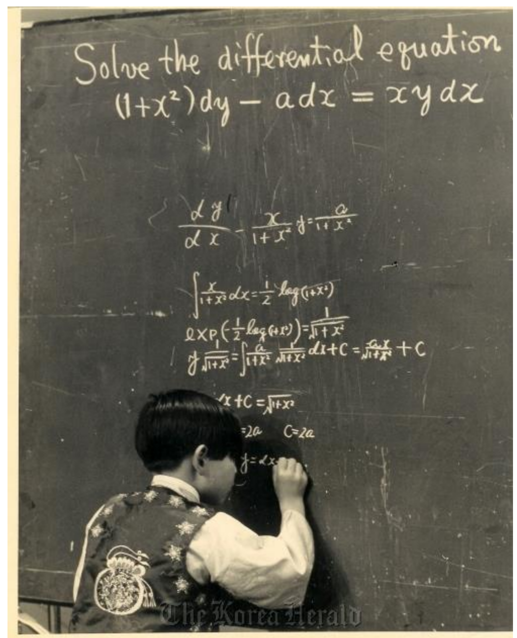


$$\frac{dy}{dx} = y - x$$

$$y = Ce^{x} + x + 1$$

Kim Ung Yong

Kim Ung Yong: Korean prodigy, born 3 March 1962



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